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Two-dimensional conformal field theories on AdS_{2d+1} backgrounds

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Abstract: Various exact two-dimensional conformal field theories with AdS_{2d+1} target space are constructed. These models can be solved using bosonization techniques. Some examples are presented that can be used in building perturbative superstring theories with AdS backgrounds, including AdS_5.

Keywords: Conformal Field Models in String Theory, String Duality, Superstring Vacua.

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Contents

1. Introduction

The dualities [1, 2, 3] between certain conformal field theories and string theories on anti-de Sitter (AdS) spaces with RR fluxes have led to renewed interest in the description of 2d conformal field theories with RR backgrounds. A useful description of the latter would allow one to go beyond the supergravity approximation and study $1/N$ and $1/g^2N$ corrections in the field theory.

The description of RR backgrounds in the NSR formulation is a difficult problem. It is not even known how to describe a condensate of RR vertex operators at the classical level (for a recent attempt, see [4]). On the other hand, the classical description of $AdS_5 \times S^5$ and $AdS_3 \times S^3$ backgrounds with RR fields in the Green-Schwarz formalism is known [5]. The quantization of the GS string is more complicated than the quantization of the NSR string. In flat space, the GS string can be quantized in light-cone gauge, which breaks manifest target space covariance. This is no longer possible on curved space like AdS; quantization in other gauges has been discussed in [6]

A third, more promising approach is to employ GS type variables for the NSR string, which combines the covariance of the GS approach with the quantizability of the NSR string. Such variables exist in four [7] and six [8] dimensions. In ten dimensions variables that preserve $\mathrm{U}(5) \subset \mathrm{SO}(10)$ super Poincaré invariance exist [9]. The six-dimensional variables can be used to find a description of superstrings on $AdS_3 \times S^3$ with both RR and NS background fields turned on [10]

The case of $AdS_3 \times S^3$ with RR background is special because the theory can be S-dualized to $AdS_3 \times S^3$ with only NS fields turned on, which in turn can be described in a straightforward way using WZW models. The resulting theory and the relation between the space-time and world-sheet CFTs were studied in [11, 12, 13]. We believe that the AdS_5 and AdS_7 CFTs described in this paper can be studied in a similar fashion.

In general, to construct a world-sheet description of a given string background one needs to find a 2d CFT that describes that background, check it has the correct central charge, and couple it to ghosts (or ghost-like variables depending on the approach). Finding exact conformal sigma models is not an easy problem. The models that have been most extensively studied and that can be solved exactly are rational conformal field theories. Many of those correspond to WZW theories (or their cosets). The corresponding Lagrangians are two-dimensional sigma models on group manifolds that include WZ terms. A priori the coefficients in front of the kinetic and the WZ term are unrelated, but for most groups conformal invariance requires them to be proportional to each other. An exception are the supergroups $\mathrm{PSL}(n|n)$, for which it has been shown recently that the theory is conformal for arbitrary values of the two parameters [10, 14]. In fact, the description in [10] of superstrings on $AdS_3 \times S^3$ is based on a sigma model with target space the group manifold $\mathrm{PSL}(2|2)$, coupled in a certain way to ghost fields. Together with the physical state condition this provides a complete world-sheet description of $AdS_3 \times S^3$ background with an arbitrary amount of RR and NS fluxes.

In this paper we construct exact CFTs with AdS_{2d+1} backgrounds and compute their central charges. As far as we know no previous examples of exact quantum CFTs with AdS_{2d+1} target space ($d > 1$) were known, independently of whether RR background fields are turned on or not. The exact CFTs we construct all correspond to a novel type of cosets of WZW models, and depend on one free parameter. Standard cosets cannot be used, because their target spaces are generically singular spaces of the form G/H where H acts on G as $g \to h^{-1}gh$. Homogeneous spaces like AdS_d, on the other hand, are of the form G/H where H acts from the left or right on G, and gauging such a subgroup in WZW models is anomalous.

Let us list some results:

1. For the purely bosonic case we claim that the sigma model

$$S_{AdS_{2d+1}} = \frac{k}{2\pi}\int d^2z(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}_r\bar{\partial}\gamma_r) + \frac{1}{2\pi}\int d^2z(d-1)\phi\sqrt{g}R\,, \quad r=1,\ldots,d \tag{1.1}$$

is exactly conformal. It has AdS_{2d+1} as target space, a non-zero B-field and dilaton and Virasoro central charge:

$$c = (2d+1) + \frac{6}{k-2d}\,. \tag{1.2}$$

The non-zero B-field depends on a choice of complex structure on the boundary of AdS_{2d+1} and therefore breaks the Lorentz group from SO$(2d)$ to U(d). It is an interesting question whether a supersymmetric version of this theory (see below) can be mapped (using a combination of T- and S-dualities) to AdS_5 with only the RR five-form turned on.

2. One can also consider the novel type of cosets for supergroups. The resulting sigma models will contain anti-commuting (world-sheet scalar) variables θ. For example, for the case of SL$(N|N)$/SL$(N-1)^2$ the resulting sigma model contains $2N^2$ real anti-commuting fields. The bosonic part of the sigma model has as metric $AdS_{2N-1}\times AdS'_{2N-1}$ with a nonzero B-field and dilaton. The prime on AdS' indicates that the metric is that of AdS but with a wrong sign for $d\phi^2$, $ds'^2 = -d\phi^2 + e^{2\phi}|d\gamma_r|^2$. Besides a kinetic term the anti-commuting variables also have a fermionic B-field. The precise meaning of this fermionic B-field depends on the way the sigma model is promoted to a full-fledged string theory (i.e. on the coupling to the ghost degrees of freedom). Although this will not be discussed in this paper, we suspect that there is a close relation between the fermionic B-field and RR background fields. The central charge of the SL$(N|N)$/SL$(N-1)^2$ sigma model is $c=-2N^2+4N-2$. For $N=3$ we get a ten-dimensional space-time with $c=-8$.

   Another interesting example is SL$(4|4)$/SP$(2)^2$. The corresponding homogeneous superspace plays an important role in the construction of the action for the GS string on $AdS_5\times S^5$. In our case we find as space-time metric $AdS_5\times AdS'_5$ (we will comment on the relation between AdS'_5 and S^5 later), and 32 anti-commuting world-sheet scalars. Furthermore, the central charge of this theory is $c=-22$. It is tempting to conjecture a relation to the GS type of approach; however, our sigma model does not appear to have an analogue of kappa symmetry.[1]

[1] We thank M. Bershadsky for discussions on the bosonization of current algebras for supergroups and related questions.

3. If we combine the examples given under 1 with their analytic continuations from ϕ to $i\phi$ (which corresponds to changing the geometry of AdS to that of AdS') we get bosonic conformal sigma models with target space $AdS_{2d_1+1} \times AdS'_{2d_2+1}$. If $d_1 + d_2 = 4$ and the two levels $k_{1,2}$ of the two individual sigma models satisfy $k_1 - 2d_1 = -(k_2 - 2d_2)$ the central charge is exactly equal to ten. By adding NSR type world-sheet fermions we can construct a theory with $N = 1$ world-sheet superconformal invariance and central charge $c = 15$. The number of unbroken target space supersymmetries equals $(d_1 + 1)(d_2 + 1)$.

   We will bosonize all models discussed above, thereby showing that they are "exactly solvable". We put the exact solvability between quotation marks because we expect to encounter the same difficulties that one meets when one tries to solve the noncompact SL$(2, R)$ WZW model using bosonization.

The outline of this paper is as follows. In section 2 we describe the bosonic sigma models with target space AdS_{2d+1} and show that the corresponding background fields solve (the lowest order in α') supergravity equations of motion. We will also discuss the global and local symmetries of the "cosets" and comment on their holographic nature.

In section 3 we describe how such models can be obtained by means of a novel kind of cosets of WZW models. This procedure was first hinted in [15, 16, 17] in the search of WZW cosets that have a quadratic stress-tensor upon bosonisation. There is no known geometric interpretation of this coset construction although we will attempt to formulate it as geometrical as possible. We use the results of [18] in order to write the classical Lagrangian for such "cosets". For the particular cases of SL(3)/SL(2) and SL(4)/SP(2) we will see the emergence of the AdS_5 metric.

In section 4 we study NSR strings in these backgrounds. We describe the bosonization including the NSR fermions and show that $N = 1$ world-sheet supersymmetry is preserved.

In section 5 we combine different NSR "cosets" to construct critical superstring backgrounds with $\hat{c} = 10$. We discuss the target space supersymmetries and comment on the GSO projection. The theory that resembles $AdS_5 \times S^5$ preserves, quite intriguingly, 18 of the 32 spacetime supersymmetries.

In section 6 we then turn to the generalization of the "cosets" to supergroups, and in particular focus on the interesting case of SL(4|4)/SP$(2)^2$. Here we will face some difficulties which are most likely related to something which has become a standard lore by now, namely that the only consistent IIB string background with maximal SUSY (32-supercharges) are given by flat space or $AdS_5 \times S^5$ (which has SO(6) global symmetry, N-units of RR five form flux, and no other fields turned on).[2] Our backgrounds have parity violating WZ terms, a reduced global symmetry

[2]We thank A. Strominger for pointing this out to us.

and therefore we do not expect to be able to recover the usual $AdS_5 \times S^5$ background, although we do get a CFT on the world-sheet whose target space metric is that of AdS_5.

We note that in the study of [10] there are two distinguished points in parameter space: 1) the WZW point — when the parameters in front of the kinetic and WZ terms are related, leading to a PSL(2|2) current algebra symmetry; 2) when the WZ term is absent. The sigma models studied in this paper that are based on supergroups are analogues of 1). In the case considered in [10] the RR flux is zero at this point. It is not clear whether this is true for the higher dimensional AdS spaces. Another interesting question that is not discussed in this paper is whether it is in general possible to perturb away from the WZW point, and if so, how.

Finally, section 7 describes various open problems and directions for future research.

2. Sigma models for AdS_{2d+1}

The simplest way to introduce the conformal field theories presented in the introduction is to start with the well-known case of the $SL(2,\mathbb{R})$ WZW model. The conformal sigma model with AdS_3 target space is given by the WZW action for $SL(2,\mathbb{R})$ in the Gauss parametrization and reads [19]

$$S = \frac{k}{2\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma \right). \tag{2.1}$$

This theory has an sl_2 current algebra. By introducing auxiliary fields $\beta, \bar{\beta}$ and rescaling ϕ the action can be rewritten as

$$S = \frac{1}{4\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-2\phi/\alpha_+} - \frac{2}{\alpha_+}\phi\sqrt{g}R \right), \tag{2.2}$$

where $\alpha_+ = \sqrt{2(k-2)}$. This latter action describes a free field ϕ with some background charge and two free β, γ systems, perturbed by the exactly marginal operator $V = -\beta\bar{\beta}e^{-2\phi/\alpha_+}$. The operator V is of the form $S\bar{S}$, with $S = \beta e^{-2\phi/\alpha_+}$ a dimension $(1,0)$ operator known as the screening current. The contour integral of S is known as the screening charge of the theory. The only holomorphic operators in the theory that commute with $\oint S$ are those constructed out of the sl_2 currents. In addition, the correlation functions of the theory can, after an appropriate number of screening charges have been inserted, be computed in the free field approximation. In fact, the results obtained a decade ago in the study of correlation functions for WZW models using the bosonisation technique for conformal blocks produced results that were only valid for compact groups; note that for SU(2) what one does is the analytic continuation $\phi \to i\phi$ — this leads to the correct correlators although the analytically continued sigma model doesn't describe anything close to the SU(2) WZW action, see for example [16] for the lagrangean approach.

To write down sigma models for AdS_{2d+1}, we simply take d copies of the single fields $\gamma, \bar{\gamma}, \beta, \bar{\beta}$, and write down the same actions as above. The background charge in (2.2) remains unchanged in order for the perturbations to be exactly marginal. However, if we integrate out the fields $\beta, \bar{\beta}$ we now generate a nonzero background charge in (2.1), because each pair of $\beta, \bar{\beta}$ contributes $+2/\alpha_+$ to the background charge. Thus, the appropriate generalizations of (2.1) and (2.2) read

$$S = \frac{k}{2\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}_r\bar{\partial}\gamma_r \right) + \frac{1}{2\pi} \int d^2z \, (d-1)\phi\sqrt{g}R \, . \tag{2.3}$$

and

$$S = \frac{1}{4\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + \beta_r\bar{\partial}\gamma_r + \bar{\beta}_r\partial\bar{\gamma}_r - \beta_r\bar{\beta}_r e^{-2\phi/\alpha_+} - \frac{2}{\alpha_+}\phi\sqrt{g}R \right) , \tag{2.4}$$

where $r = 1, \ldots, d$ is summed over, and $\alpha_+ = \sqrt{2(k-2d)}$.

In (2.3) we recognize a standard sigma model on AdS_{2d+1} with nonzero B field and linear dilaton. It may appear somewhat surprising that these background fields solve (the lowest order in α') supergravity equations of motion. We will now show that this is indeed the case.

Recall that supergravity equations of motion with only a nonzero metric, NS two-form and dilaton read

$$\begin{aligned} R_{ij} + \frac{1}{4H_{ikl}H_j{}^{kl}} - 2\nabla_i\nabla_j\Phi &= 0 \\ \nabla^k H_{kij} - 2\nabla^k\Phi H_{kij} &= 0 \, . \end{aligned} \tag{2.5}$$

For the metric we take the AdS_{2d+1} metric

$$ds^2 = d\phi^2 + e^{2\phi}(d\gamma_r d\bar{\gamma}_r) \tag{2.6}$$

and for the two form

$$B = e^{2\phi} d\gamma_r \wedge d\bar{\gamma}_r \, , \tag{2.7}$$

where $r = 1, \ldots, d$. In addition, we take the linear dilaton from (2.3),

$$\Phi = (d-1)\phi \, . \tag{2.8}$$

It is now straightforward to compute

$$R_{ij} = 2d g_{ij} \tag{2.9}$$

and since $H_{\phi\gamma_r\bar{\gamma}_r} = e^{2\phi}$, we get

$$H_{\phi kl}H_\phi{}^{kl} = -8d g_{\phi\phi} \, , \qquad H_{\gamma_r kl}H_{\bar{\gamma}_r}{}^{kl} = -8 g_{\gamma_r\bar{\gamma}_r} \, . \tag{2.10}$$

Furthermore, we find that

$$\nabla_{\gamma_r}\nabla_{\bar{\gamma}_r}\Phi = (d-1)g_{\gamma_r\bar{\gamma}_r}\,, \qquad \nabla^k H_{k\gamma_r\bar{\gamma}_r} = 2(d-1)e^{2\phi}\,. \tag{2.11}$$

Thus, both supergravity equations of motion are satisfied.

The one-loop correction to the central charge of the theory is given by

$$c = (2d+1) + 3\alpha'\left(4\nabla_i\Phi\nabla^i\Phi - 4\nabla^2\Phi + R + \frac{1}{12}H_{ijk}H^{ijk}\right) \tag{2.12}$$

and we find

$$c = (2d+1) + 12\alpha' + \mathcal{O}(\alpha'^2)\,. \tag{2.13}$$

The usual coefficient in front of (2.3) is $(4\pi\alpha')^{-1}$, so that $\alpha' \sim (2k)^{-1}$. We can also compute the central charge to all orders using the free field representation (2.4), with the result

$$c = (2d+1) + \frac{6}{k-2d} \tag{2.14}$$

which agrees with (2.13) up to terms of order k^{-2}.

2.1 Holographic aspects

If we approach the boundary of AdS_{2d+1}, the description in terms of free fields given by (2.4) becomes very useful. Both the string coupling and the perturbation become very small near the boundary. On the other hand, it seems that the space-time theory is strongly coupled near the boundary, because the string coupling goes to infinity. However, the growth of the strength of the string interactions has to compete against the rate at which points at fixed $\gamma, \bar{\gamma}$ separate near the boundary. The situation has some similarities to the one discussed in [20]. To analyze these competing effects, it is better to pass to the Einstein frame. In the Einstein frame, the metric on AdS_{2d+1} becomes

$$ds_E^2 = e^{2\phi/(2d-1)}(e^{-2\phi}d\phi^2 + d\gamma_r d\bar{\gamma}_r)\,. \tag{2.15}$$

Because of the positive power of e^ϕ in front of this expression, distances on the boundary parametrized by $\gamma_r, \bar{\gamma}_r$ go to infinity as ϕ goes to infinity, and therefore we expect holographic behavior for large ϕ as in [20].

A useful set of vertex operators in the theory is given by wave functions which are solutions of the Laplace equation on AdS_{2d+1} that behave as a delta function near a given boundary point. As in [12, 13] correlation functions of such operators will compute correlation functions of the (unknown) theory living on the boundary.

2.2 Reality conditions

So far we have not yet specified which reality conditions the fields satisfy. This depends on the precise AdS space we wish to consider. Euclidean AdS_{2d+1} is given by the equation

$$\sum_{r=1}^{d}(x_r^2 + y_r^2) + z^2 - w^2 = -1 \tag{2.16}$$

in the space with metric

$$ds^2 = \sum_{r=1}^{d}(dx_r^2 + dy_r^2) + dz^2 - dw^2 \,. \tag{2.17}$$

We parametrize solutions to (2.16) as follows

$$\begin{aligned} x_r + iy_r &= \gamma_r e^{\phi}\,, & x_r - iy_r &= \bar{\gamma}_r e^{\phi}\,, \\ z + w &= e^{-\phi} + e^{\phi}\textstyle\sum_r \gamma_r\bar{\gamma}_r\,, & z - w &= -e^{-\phi}\,. \end{aligned} \tag{2.18}$$

If we insert this into (2.17) we obtain the AdS_{2d+1} metric in the form $ds^2 = d\phi^2 + e^{2\phi}d\gamma_r d\bar{\gamma}_r$. Thus, to describe Euclidean AdS we require that γ_r and $\bar{\gamma}_r$ are each others complex conjugate. Other signature AdS spaces are obtained by taking γ_r and $\bar{\gamma}_r$ real and independent for certain r, or by taking them to be minus each others complex conjugate. Lorentzian signature AdS has one pair of real independent γ, $\bar{\gamma}$. All possible signature AdS spaces can be obtained this way, except S^{2d+1}.

It is at present not clear to us whether there exists a generalization of the AdS_{2d+1} sigma models to S^{2d+1} for $d > 1$. For $d = 1$, as we already mentioned, there is a close relation between the $\mathrm{SL}(2,\mathbb{R})$ WZW theory at negative level and the $\mathrm{SU}(2) = S^3$ WZW theory at positive level which in terms of the variables in (2.1) requires analytic continuation in ϕ. Perhaps a similar analytic continuation of AdS_{2d+1} to negative level has an interpretation as a CFT on S^{2d+1}. Because we don't have full understanding in the simplest case of $\mathrm{SL}(2,\mathbb{R}) \leftrightarrow \mathrm{SU}(2)$, but analytic continuation does give correct results, we will assume that a similar procedure works for the $AdS_{2d+1} \leftrightarrow S^{2d+1}$ case as well. It would be interesting to investigate this further.

2.3 Global and local symmetries

The global symmetries of (2.3), ignoring the dilaton, are for $d > 1$ given by

$$\delta\phi = c\,, \qquad \delta\gamma_r = a_r + b_{rs}\gamma_s\,, \qquad \delta\bar{\gamma}_r = \bar{a}_r + \bar{b}_{rs}\bar{\gamma}_s\,, \tag{2.19}$$

with the parameters subject to the condition

$$b_{rs} + \bar{b}_{sr} + 2c\delta_{rs} = 0\,. \tag{2.20}$$

The reality conditions on the parameters follow from those on the fields. The number of global symmetries is equal to $(d+1)^2$. For Euclidean AdS_{2d+1} the group of global symmetries contains an obvious $\mathrm{U}(d)$ group of global symmetries. The full group is smaller than the isometry group of AdS_{2d+1} which is $\mathrm{SO}(2d+1,1)$. The group of global symmetries is broken by the presence of the NS two-form, whose field strength in terms of the coordinates (2.16) is proportional to $H \sim (z-w)^{-1}d(z-w)\wedge dx_r\wedge dy_r$.

For $d = 1$ the global symmetries of euclidean AdS_3 are $\mathrm{SL}(2,\mathbb{C})$, for lorentzian AdS_3 $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Besides the global symmetries there are also holomorphic symmetries of AdS_{2d+1}, that give rise to holomorphic currents. These currents are given by

$$J_r(z) = e^{2\phi}\partial\bar{\gamma}_r = \beta_r\,, \qquad J_0(z) = -\partial\phi + e^{2\phi}\sum_r \gamma_r\partial\bar{\gamma}_r = \sum_r \beta_r\gamma_r - \alpha_+\partial\phi \tag{2.21}$$

and remain holomorphic in the full quantum theory (via the second equality above). These operators commute with the screening charges:

$$Q_r = \int dz \beta_r e^{-\frac{2}{\alpha_+}\phi} \tag{2.22}$$

and also form the OPE algebra:

$$\begin{aligned} J_r(z)J_{r'}(w) &= 0\,, \\ J_r(z)J_0(w) &= \frac{-2J_r(z)}{z-w}\,, \\ J_0(z)J_0(w) &= \frac{-2k}{(z-w)^2}\,. \end{aligned} \tag{2.23}$$

In order to determine the complete chiral algebra one should find all polynomials in the free fields and their derivatives that commute with the screening charges. One such operator is the stress-tensor:

$$T(z) = -\frac{1}{2}\left(\sum_r \beta_r\partial\gamma_r + (\partial\phi)^2 + \frac{2}{\alpha_+}\partial^2\phi\right)\,. \tag{2.24}$$

3. Bosonic WZW cosets

It was noticed in [15, 16, 17] that given a group G and a subgroup $H = H_1 \times H_2 \times \cdots$, such that the levels of the corresponding current algebras satisfy $k^G + c_V^G = k^{H_1} + c_V^{H_1} = \cdots$, one can attempt to define some sort of "coset" theory, by getting rid of the degrees of freedom residing in H. The basic idea of this construction is to do a partial bosonization of the G currents, by expressing them in terms of the H currents and extra β, γ, ϕ systems, and to subsequently set the H currents to zero. One finds that the stress tensor that survives is quadratic in the remaining fields. For a special value of the level k^G, the original G current algebra survives. General expressions for the partial bosonizations that one needs to implement this construction are given in [18]. We first illustrate this procedure for the case SL(3)/SL(2) in the Lagrangian formalism, and then turn to more general examples.

3.1 SL(3)/SL(2)

The discussion in this section will be completely classical. To discuss the coset for SL(3)/SL(2), we start by taking the following Gauss decomposition for SL(3)

$$G = \begin{pmatrix} 1 & \bar{\gamma}_1 & \bar{\gamma}_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g_{sl_2} \begin{pmatrix} e^{-2\phi} & 0 & 0 \\ 0 & e^{\phi} & 0 \\ 0 & 0 & e^{\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

where g_{sl_2} is a group element of SL(2), which is embedded in sl_3 in the bottom right 2×2 block. We write $G = g_U g_{sl_2} g_D g_L$. The WZW action evaluated for G is

$$S_{WZW}(G) = S_{WZW}(g_D) + S_{WZW}(g_{sl_2}) + \frac{k}{2\pi} \int \mathrm{tr}\left(g_U^{-1} \partial g_U g_{sl_2} g_D \bar{\partial} g_L g_L^{-1} g_D^{-1} g_{sl_2}^{-1} \right). \quad (3.2)$$

The last term above can be written explicitly:

$$e^{3\phi} \partial \bar{\gamma}_i g_{sl_2}^{ij} \bar{\partial} \gamma_j \,. \quad (3.3)$$

This form shows the interaction between SL(2) and remaining degrees of freedom. The idea of the novel type of coset is to set $g_{sl_2} = 1$. Classically this is not problematic, but in the quantum theory g_{sl_2} has a nonzero conformal weight and cannot simply be put equal to one. Later we will see how the theory has to be modified so as to preserve conformal invariance after g_{sl_2} has been put equal to one. After putting g_{sl_2} equal to one, the target space M of the theory will be the submanifold $g_U g_D g_L \subset G$ parametrized by g_U, g_D and g_L. This is not quite the homogeneous space SL(3)/SL(2), because tangent vectors to M are not orthogonal to the sl_2 Killing vectors, and the corresponding "Faddeev-Popov" determinant is absent. It is also not the target space of the usual coset construction. In fact, as we will see below, it is AdS_5.

The next step is to bosonize (3.2) in the lagrangean approach, following [16], which involves changing variables from $\bar{\gamma}_i$ and γ_i to β_i and γ_i, so that the last term in (3.2) becomes the free action $\int \beta_i \bar{\partial} \gamma_i$. Therefore, the change of variables reads

$$\beta_i = e^{3\phi} g_{sl_2} \partial \bar{\gamma}_i. \quad (3.4)$$

This change of variables has a jacobian which can be found for example in [16]. This jacobian leads to a shift in the coefficient of the kinetic term for ϕ, introduces an improvement term for ϕ (i.e. a linear dilaton background) and changes the level of the sl_2 WZW theory, all in such a way that the conformal invariance and central charge of the theory remain unaltered. However, the discussion in this section will be purely classical and we will ignore this jacobian.

In matrix form, the change of variables (3.4) reads

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = g_D^{-1} g_{sl_2}^{-1} g_U^{-1} \partial g_U g_{sl_2} g_D \quad (3.5)$$

When inserted in the action we find that the last term becomes the sum of two free beta-gamma systems, and it is also immediately clear that the stress-energy tensor will be the direct sum of that of the WZW theories for g_D, g_{sl_2} and that of the two beta-gamma systems. By working out the sl_3 current $G^{-1}\partial G$, we find an expression for the sl_3 current in terms of the sl_2 currents, and $\gamma_1, \gamma_2, \beta_1, \beta_2, \phi$. This is the realization as in (4.7.22) of [16]. In addition, (3.5) enables us to express $\partial\gamma_1$ and $\partial\gamma_2$ in terms of the other fields. Requiring that the integral of these vanish, or are equal to the monodromies of γ, provides the two screening charges for this type of free field realization. (The number of screening charges is always $\dim g_L = \dim g_U$.) Now let us try to remove the sl_2 degrees of freedom by putting $g_{sl_2} = 1$. The relation (3.5) with $g_{sl_2} = 1$ becomes

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = g_D^{-1} g_U^{-1} \partial g_U g_D = \begin{pmatrix} 0 & e^{3\phi}\partial\bar{\gamma}_1 & e^{3\phi}\partial\bar{\gamma}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6}$$

The screening charges are therefore $\int e^{-3\phi}\beta_1$ and $\int e^{-3\phi}\beta_2$, similar to (4.1.22-24) of [16] with the sl_2 degrees of freedom removed. We now want to use these screening charges to find the action of the reduced theory. Thus we substitute $\beta_1 = e^{3\phi}\partial\bar{\gamma}_1$ and $\beta_2 = e^{3\phi}\partial\bar{\gamma}_2$ back into the action

$$S = S_{WZW}(g_D) + \frac{k}{2\pi}\int (\beta_1\bar{\partial}\gamma_1 + \beta_2\bar{\partial}\gamma_2) \tag{3.7}$$

and get

$$S = \frac{1}{2\pi}\int \left(3\partial\phi\bar{\partial}\phi + e^{3\phi}(\partial\bar{\gamma}_1\bar{\partial}\gamma_1 + \partial\bar{\gamma}_2\bar{\partial}\gamma_2)\right) . \tag{3.8}$$

Up to a rescaling, this is the action of AdS_5 that we presented in the previous section. The final action in terms of group variables simply reads

$$S = S_{WZW}(g_U g_D g_L) . \tag{3.9}$$

Thus the process of first bosonizing, removing the sl_2 degrees of freedom and then undoing the bosonization amounts to the same as removing the sl_2 degrees of freedom from the Gauss decomposition of the SL(3) group element. In fact, as will we argue later, the action (3.9) describes an exact conformal field theory once a suitable linear dilaton background, which makes the conformal weights of all screening currents equal the one, is included.

3.2 Bosonization

Let us now discuss the bosonization of the sigma model for AdS_{2d+1}, given in (2.3), in the path integral framework. The action reads

$$S = \frac{k}{2\pi}\int d^2z\,(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}_r\bar{\partial}\gamma_r) + \frac{1}{2\pi}\int d^2z\,(d-1)\phi\sqrt{g}R . \tag{3.10}$$

The measure in the path integral is defined through the sigma model metric as

$$\int \left[(\delta\phi)^2 + e^{2\phi}\sum \delta\gamma_r\delta\bar{\gamma}_r\right] . \tag{3.11}$$

We can map the sigma model to a theory of free fields if we introduce new fields

$$\beta_r = e^{2\phi}\partial\bar{\gamma}_r \,. \tag{3.12}$$

The jacobian J for this change of variables is nontrivial and can be computed through the anomaly in the determinant

$$J = \det{}^{d}\left[e^{-2\phi}\bar{\partial}e^{2\phi}\partial\right] \tag{3.13}$$

exactly like in [16] leading to the result

$$\log J = \frac{d}{(4\pi)}\int d^2z\left[4\partial\phi\bar{\partial}\phi + 2R\phi\right]. \tag{3.14}$$

At the same time after the above change of variables the measure becomes the standard one for the free fields β_r, γ_r, ϕ and finally we get the quantum action

$$S_q = \frac{1}{4\pi}\int d^2z\left((2k-4d)\partial\phi\bar{\partial}\phi + 2k\beta_r\bar{\partial}\gamma_r - 2\phi\sqrt{g}R\right). \tag{3.15}$$

The change of variables also produces the screening charges

$$\int \partial\bar{\gamma}_r = \int \beta_r e^{-2\phi} \,. \tag{3.16}$$

The background charge of the field ϕ in (3.15) is precisely such that the screening charge has conformal weight zero, which is a necessary condition for conformal invariance.

At this stage we have obtained two bosonized version of the sigma model (2.3), namely the chirally bosonized version (3.15) and the non-chirally bosonized version (2.4). One of the main applications of free field realization for WZW models is that it allows one to compute correlator functions. This has been worked out in [15, 16, 21, 22, 23, 24, 25, 26].

The chiral bosonized version of the action can be used to compute the conformal blocks of the CFT. To combine left and right-moving conformal blocks we need the non-chirally bosonized action. The non-chirally bosonized action consists of the chiral and anti-chiral bosonized action, perturbed by operators of the form $S\bar{S}$ where S is often a screening current. Suppose that we are interested in studying a CFT with some chiral algebra A. If A is generated by the operators that commute with a given set of screening charges $Q_r = \oint S_r$, then the conformal blocks of the theory can be studied using only the screening charges Q_r, which are called the simple screening charges. One might think that the action describing the full CFT is then simply the sum of the chiral and anti-chiral bosonized action perturbed by $\sum S_r\bar{S}_r$. This is not the full story however. The variation of S_r under the transformations generated by an element of A is a total derivative, $\delta S_r = \partial Y_r$, so that the screening charges

Q_r are invariant. It is then clear that the variation of $\int d^2z \sum_r S_r\bar{S}_r$ is not zero but rather $-\int d^2z \sum_r Y_r\partial\bar{S}_r$. This vanishes on-shell but not off-shell. To find an action which is invariant, we need to modify the transformation rules of the fields and sometimes include additional terms in the action that look like higher powers of the simple screening currents, and which give rise to the non-simple screening charges. This does not happen in (2.4), but if we bosonize the SL(3) WZW theory we find that the free field action perturbed by the two simple screening currents does not have SL(3) current algebra; for that one needs to add an additional term to the action, which is built out of iterated simple screening currents, i.e. non-simple screening currents. After the addition of the additional term, the action is identical to the classical SL(3) WZW action. This procedure is certainly correct for SL(N), but we are not aware of any good explanation in the literature. (As we mentioned before, bosonization allows one to exactly solve WZW theories with compact groups like SU(N), by analytic continuation of the results for SL(N). The precise meaning of holomorphic factorization for noncompact groups like SL$(2,\mathbb{R})$ is not clear; one expect effects similar to the Liouville model, where one finds that correlators depend non-analytically on the cosmological constant μ, which is the coefficient in front of the perturbation e^{ϕ} in the lagrangean.) The point we want to make is that in order to compute certain quantities it is sufficient to give a set of simple screening charges and the symmetries of the problem one wants to preserve. The simple screening charges give the conformal blocks, and the coupling of left and right movers follows by imposing the symmetries of the problem. The symmetries of the problem allow one to construct, order by order, the action that is invariant under those symmetries, which then in turns provides the required coupling of left and right movers. An example of this will appear in section 4 of the paper.

3.3 General bosonic case

The type of coset construction we have been considering can in principle be done for any generalized Gauss decomposition $g = g_u h_0 g_d g_l$. Here, h_0 takes values in a product of simple factors $h_0 \in H \equiv H_1 \times H_2 \times \cdots$, g_d takes values in an abelian group, and g_d and g_l are certain upper and lower triangular matrices. The idea of the coset construction is to somehow get rid of the degrees of freedom residing in H [16].

In case the subgroup $G_u H G_d$ of G is parabolic, the construction of [18] can be used to find an explicit representation of the G currents in terms of β, γ systems associated to $G_{u,l}$, scalar fields ϕ_i parametrizing G_d, and $H = H_1 \times H_2 \cdots$ currents. The level k_i of the H_i currents is $k_i + c_V^{H_i} = k^G + c_V^G$. In addition, there is a set of screening charges, and the stress tensor of the theory contains improvement terms for the scalars ϕ_i. To get rid of the degrees of freedom in H we cannot use conventional coset techniques, because the H currents themselves are not part of the theory (they do not commute with the screening charges). This is reflected in the fact that the level of the H_i current algebra differs from that of G. Therefore we propose to do the following: remove the H degrees of freedom by hand, and subsequently adjust the background charges in such a way that the screening charges keep the right conformal weight. Classically this amounts to replacing $S_{\rm WZW}(g_u h_0 g_d g_l)$ by $S_{\rm WZW}(g_u g_d g_l)$, but in the quantum theory we will also find a nontrivial dilaton. Because the screening charges give exactly marginal deformations of the reduced theory, this procedure will always give an exact conformal background. Its correlation functions can in principle be computed using free field methods. Nevertheless, it would be interesting to know if there is a more direct way to obtain the reduced theory from the original theory. For instance, one could try to gauge the H-degrees of freedom, thereby replacing the H-WZW theory by a topological H/H model, or one could try to define the reduced theory as the BRST cohomology of a suitable BRST operator of the original theory.

We illustrate the general method now by considering the case $G = \mathrm{SL}(N)$ and $H = \mathrm{SL}(p_1) \times \cdots \times \mathrm{SL}(p_r)$, with $\sum p_i = N$. The $\mathrm{SL}(p_i)$ are embedded as diagonal $p_i \times p_i$ matrices in $\mathrm{SL}(N)$, with $\mathrm{SL}(p_1)$ appearing in the top left block, and $\mathrm{SL}(p_r)$ appearing in the right bottom block. The abelian group G_d is parametrized by $r-1$ scalar fields ϕ_i. In addition there are $r-1$ screening charges, associated to the simple roots $\alpha_{p_1}, \alpha_{p_1+p_2}, \ldots, \alpha_{p_1+\ldots+p_{r-1}}$. The $i^{\rm th}$ screening charge is composed out of polynomials in the β,γ-fields, a vertex operator for the H-theory that creates the $p_i p_{i+1}$ dimensional (bifundamental) representation of $\mathrm{SL}(p_i) \times \mathrm{SL}(p_{i+1})$, and an exponential of the scalar fields ϕ_i. If we now remove the H degrees of freedom from the theory, the $i^{\rm th}$ screening charge decomposes into $p_i p_{i+1}$ screening charges, each with the same exponent of ϕ_i. To make sure that all screening charges are contour integrals of objects of conformal weight one, we need to adjust the background charges of the scalar fields ϕ_i. Since there are $r-1$ different exponentials of the fields ϕ_i, we get $r-1$ equations for $r-1$ background charges, that have a unique solution. The stress tensor of the reduced theory is then the sum of free β,γ stress tensors and improved stress tensors for ϕ_i. The central charge of this theory can be shown to be equal to

$$c = \dim G_d + \dim G_u + \dim G_l + \frac{3y}{k - c_V^G}\,, \tag{3.17}$$

where

$$\begin{aligned} y = {} & \sum_{i=1}^{r-1} \left(\frac{1}{p_i} + \frac{1}{p_{i+1}}\right)^2 \frac{(p_{i+1} + \cdots + p_N)(p_1 + \cdots + p_i)}{N} + \\ & +2 \sum_{1 \le i < j \le r-1} \left(\frac{1}{p_i} + \frac{1}{p_{i+1}}\right)\left(\frac{1}{p_j} + \frac{1}{p_{j+1}}\right) \frac{(p_{j+1} + \cdots + p_N)(p_1 + \cdots + p_i)}{N}\,. \end{aligned} \tag{3.18}$$

As a check, we notice that for $p_i = 1$ and $r = N$, we get $y = N(N^2-1)/3$ and we recover from (3.17) the usual central charge of the $\mathrm{SL}(N)$ WZW theory. In this case the reduced theory is equivalent to the original WZW theory, as H is trivial.

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Another example that is straightforward is SL$(d+1)$/SL(d). This corresponds to $r=2$ and $p_1=1, p_2=d$. We get $y=(d+1)/d$ and

$$c=(2d+1)+\frac{3(d+1)}{d(k-d-1)}\,. \tag{3.19}$$

The resulting theory describes strings propagating on AdS_{2d+1}. To see this, we generalize the example described in the beginning of section 2.1 to SL$(d+1)$, by taking $G_d=\mathrm{diag}(e^{-d\phi}, e^{\phi}, \ldots, e^{\phi})$. We obtain the classical action

$$S=\frac{k}{2\pi}\int d^2z\left(d(d+1)\partial\phi\bar\partial\phi+2e^{(d+1)\phi}\sum_r\partial\bar\gamma_r\bar\partial\gamma_r\right)\,. \tag{3.20}$$

After an appropriate rescaling of the fields we obtain the action as in (2.3) but with k replaced by $2kd/(d+1)$. If we make the same replacement for k in (2.14), we obtain precisely (3.19), as expected.

4. NSR formulation

We now turn to the supersymmetrization of the AdS_{2d+1} sigma models of section 2, so that they will have $N=1$ world-sheet supersymmetry. To achieve this, we write the sigma models of section 2 in terms of the following superfields

$$\begin{aligned}
\Phi &= \phi+\theta\lambda^L+\bar\theta\lambda^R+\theta\bar\theta F\\
\Gamma_i &= \gamma_i+\theta\psi_i^L+\bar\theta\psi_i^R+\theta\bar\theta G_i\\
\bar\Gamma_i &= \bar\gamma_i+\theta\bar\psi_i^L+\bar\theta\bar\psi_i^R+\theta\bar\theta\bar G_i\\
S_i &= \sigma_i+\theta\beta_i+\bar\theta u_i+\theta\bar\theta H_i\\
\bar S_i &= \bar\sigma_i+\theta\bar u_i+\bar\theta\bar\beta_i+\theta\bar\theta\bar H_i\,.
\end{aligned} \tag{4.1}$$

The supersymmetric versions of (2.3) and (2.4) read

$$S=\frac{\alpha_+^2}{4\pi}\int d^2z d^2\theta\left(D_+\Phi D_-\Phi+e^{2\Phi}D_+\bar\Gamma_i D_-\Gamma_i\right) \tag{4.2}$$

and

$$S=\frac{\alpha_+^2}{4\pi}\int d^2z d^2\theta\left(D_+\Phi D_-\Phi-\bar S_i D_+\bar\Gamma_i+S_i D_-\Gamma_i-e^{-2\Phi}S_i\bar S_i\right)\,, \tag{4.3}$$

where $D_+=\partial_\theta+\theta\partial$ and $D_-=\partial_{\bar\theta}+\bar\theta\bar\partial$. After integrating out the superfields $S_i, \bar S_i$ from (4.3) we clearly recover (4.2). In addition, there is a linear coupling of Φ to the $N=1$ super world-sheet curvature [27] that we have not explicitly written

Next, we study these actions in components. From action (4.2) we get

$$\begin{aligned}
\frac{\alpha_+^2}{4\pi}\int\Big(&-\lambda^L\bar\partial\lambda^L+\partial\phi\bar\partial\phi+F^2+\partial\lambda^R\lambda^R+\\
&+e^{2\phi}\left(-\bar\psi_i^L\bar\partial\psi_i^L+\partial\bar\gamma_i\bar\partial\gamma_i-G_i\bar G_i+\partial\bar\psi_i^R\psi_i^R\right)+2e^{2\phi}\lambda^L\left(-\bar G_i\psi_i^R+\bar\psi_i^L\bar\partial\gamma_i\right)+\\
&+2e^{2\phi}\lambda^R\left(-\partial\bar\gamma_i\psi_i^R-\bar\psi_i^L G_i\right)+e^{2\phi}(2F-4\lambda^L\lambda^R)\bar\psi_i^L\psi_i^R\Big)\,.
\end{aligned} \tag{4.4}$$

The auxiliary fields should be eliminated. They are given by

$$\begin{aligned} F &= -e^{2\phi}\bar{\psi}_i^L\psi_i^R \\ G_i &= -2\lambda^L\psi_i^R \\ \bar{G}_i &= -2\lambda^R\bar{\psi}_i^L \end{aligned} \tag{4.5}$$

and when plugged back into (4.4) we finally get

$$\begin{aligned} \frac{\alpha_+^2}{4\pi}\int\Big(&- \lambda^L\bar{\partial}\lambda^L + \partial\phi\bar{\partial}\phi + \partial\lambda^R\lambda^R + e^{2\phi}\left(-\bar{\psi}_i^L\bar{\partial}\psi_i^L + \partial\bar{\gamma}_i\bar{\partial}\gamma_i + \partial\bar{\psi}_i^R\psi_i^R\right) + \\ &+ 2e^{2\phi}\left(\lambda^L\bar{\psi}_i^L\bar{\partial}\gamma_i - \lambda^R\partial\bar{\gamma}_i\psi_i^R\right) - e^{4\phi}\bar{\psi}_i^L\psi_i^R\bar{\psi}_j^L\psi_j^R\Big)\,. \end{aligned} \tag{4.6}$$

The third line contains some terms that make the fermion kinetic terms covariant, the last line is the four-fermion term that multiplies the curvature of the connection with torsion. This curvature is not supposed to be zero unless we are in AdS_3. Indeed, the last line vanishes for AdS_3. This is because the beta-functions state that the Ricci tensor with torsion is equal to the second covariant derivative of the dilaton. So the nonvanishing four fermion term is directly related to the presence of a nonzero dilaton.

To bosonize the action (4.6) and to write it as a theory of free fields perturbed by exactly marginal operators, it is easier to start with (4.3) rather than trying to bosonize (4.6) directly.

The component expansion of (4.3) reads

$$\begin{aligned} \frac{\alpha_+^2}{4\pi}\int\Big(&- \lambda^L\bar{\partial}\lambda^L + \partial\phi\bar{\partial}\phi + F^2 + \partial\lambda^R\lambda^R - \bar{\sigma}_i\partial\bar{\psi}_i^R - \bar{u}_i\bar{G}_i + \bar{\beta}_i\partial\bar{\gamma}_i - \bar{H}_i\bar{\psi}_i^L - \\ &- \sigma_i\bar{\partial}\psi_i^L + \beta_i\bar{\partial}\gamma_i - u_iG_i + H_i\psi_i^R - e^{-2\phi}\left(\sigma_iH_i + \beta_i\bar{\beta}_i - u_i\bar{u}_i + H_i\bar{\sigma}_i\right) - \\ &- 2e^{-2\phi}\lambda^L(u_i\bar{\sigma}_i - \sigma_i\bar{\beta}_i) + 2e^{-2\phi}\lambda^R(\beta_i\bar{\sigma}_i - \sigma_i\bar{u}_i) + e^{-2\phi}(2F + 4\lambda^L\lambda^R)\sigma_i\bar{\sigma}_i\Big)\,. \end{aligned} \tag{4.7}$$

The role of $e^{2\phi}\psi_i^R$ and $e^{2\phi}\bar{\psi}_i^L$ is taken over by $\bar{\sigma}_i$ and σ_i respectively. This is clear from the H_i and $\bar{H}_i$ equations of motion. After eliminating H, G, u from (4.7) and a rescaling of the fields we get the following form of the action

$$\begin{aligned} \frac{1}{4\pi}\int\Bigg(&- \lambda^L\bar{\partial}\lambda^L + \partial\phi\bar{\partial}\phi + \partial\lambda^R\lambda^R - \bar{\sigma}_i\partial\bar{\psi}_i^R + \bar{\beta}_i\partial\bar{\gamma}_i - \sigma_i\bar{\partial}\psi_i^L + \beta_i\bar{\partial}\gamma_i - \\ &- e^{-2\phi/\alpha_+}\left(\bar{\beta}_i - \frac{2}{\alpha_+}\lambda^R\bar{\sigma}_i\right)\left(\beta_i - \frac{2}{\alpha_+}\lambda^L\sigma_i\right) - e^{-4\phi/\alpha_+}\sigma_i\bar{\sigma}_i\sigma_j\bar{\sigma}_j\Bigg)\,. \end{aligned} \tag{4.8}$$

This is probably the most useful form of the action. It has the structure discussed in section 3.2. There are d simple screening charges given by

$$S_i = e^{-2\phi/\alpha_+}\left(\beta_i - \frac{2}{\alpha_+}\lambda^L\sigma_i\right), \tag{4.9}$$

but the action is not just the sum of a free field piece and $S_i\bar{S}_i$, there is also the term $e^{-4\phi/\alpha_+}\sigma_i\bar{\sigma}_i\sigma_j\bar{\sigma}_j$. As is clear from the prefactor $e^{-4\phi/\alpha_+}$, this term is of higher order in the $e^{-2\phi/\alpha_+}$ expansion and it is needed to restore the $N=1$ world-sheet invariance of the theory. According to our discussion in section 3.2, it will not play a role in the construction of the conformal blocks of the theory, but is important in order to correctly combine left and right movers. A further clarification of this point is certainly desirable.

As a side remark, we notice that it is possible to write everything as free field theory perturbed by simple screening currents only. To accomplish this, we first add to (4.8)

$$\frac{1}{4\pi}\int d^2z\,\left(A_{ij}+e^{-2\phi/\alpha_+}\sigma_i\sigma_j\right)\left(\bar{A}_{ij}+e^{-2\phi/\alpha_+}\bar{\sigma}_i\bar{\sigma}_j\right), \tag{4.10}$$

which removes the four fermion terms but at the cost of introducing $d(d-1)/2$ auxiliary fields A_{ij}. Next we make the change of variables $A_{ij}=\partial\rho_{ij}$, $\bar{A}_{ij}=\bar{\partial}\bar{\rho}_{ij}$, and represent the corresponding determinant by an integral over $d(d-1)/2$ pairs of fermionic $\beta^{\theta}_{ij},\theta_{ij},\bar{\beta}^{\bar{\theta}}_{ij},\bar{\theta}_{ij}$ systems. Altogether, the four-fermion terms have now been replaced by the sum S_f+S_s of the free field action

$$S_f=\frac{1}{4\pi}\int d^2z\,\left(\beta^{\theta}_{ij}\bar{\partial}\theta_{ij}+\bar{\beta}^{\bar{\theta}}_{ij}\partial\bar{\theta}_{ij}+\partial\rho_{ij}\bar{\partial}\bar{\rho}_{ij}\right) \tag{4.11}$$

and the perturbation by simple screening currents

$$S_s=\frac{1}{4\pi}\int d^2z\,\left(e^{-2\phi/\alpha_+}\bar{\partial}\bar{\rho}_{ij}\sigma_i\sigma_j+e^{-2\phi/\alpha_+}\partial\rho_{ij}\bar{\sigma}_i\bar{\sigma}_j\right). \tag{4.12}$$

It is not clear, however, what replaces the $N=1$ algebra in this formulation of the theory, which is somewhat reminiscent of the formulation in [10] .

We now discuss the $N=1$ world-sheet superconformal algebra of (4.8). The $N=1$ super stress-energy tensor of (4.2) is

$$\alpha_+^{-2}G=\partial\Phi D_+\Phi-e^{2\Phi}D_+\Phi D_+\Gamma_iD_+\bar{\Gamma}_i+\frac{1}{2}e^{2\Phi}\partial\Gamma_iD_+\bar{\Gamma}_i+\frac{1}{2}e^{2\Phi}D_+\Gamma_i\partial\bar{\Gamma}_i \tag{4.13}$$

and that of (4.3) is

$$\alpha_+^{-2}G=\partial\Phi D_+\Phi+\frac{1}{2}D_+S_iD_+\Gamma_i+\frac{1}{2}S_i\partial\Gamma_i\,. \tag{4.14}$$

This does not yet contain the improvement term due to the dilaton. The improvement term is simply given by

$$G_{dil}=QD_+\partial\Phi\,. \tag{4.15}$$

Let us now focus on (4.14) and work it out in components. This will then give G and T for (4.8). We get

$$\begin{aligned}G&=\partial\phi\lambda^L+\frac{1}{2}\beta_i\psi_i^L+\frac{1}{2}\sigma_i\partial\gamma_i+Q\partial\lambda^L\\ T&=-\frac{1}{2}\left(\partial\lambda^L\lambda^L+\partial\phi\partial\phi+\frac{1}{2}\partial\sigma_i\psi_i^L+\beta_i\partial\gamma_i-\frac{1}{2}\sigma_i\partial\psi_i^L+Q\partial^2\phi\right).\end{aligned} \tag{4.16}$$

These are the suitably normalized generators of the $N = 1$ superconformal algebra with central charge $c = 3(2d+1)/2 + 3Q^2$. Their form agrees with the standard form of the $N = 1$ superconformal algebra for free fields with some improvement terms.

We need to verify whether the simple screening charges (4.9) indeed commute with the $N = 1$ algebra. From the action (4.8) we obtain the following free field OPE's

$$\begin{aligned} \phi(z)\phi(w) &\sim -\log|z-w|^2\,, & \lambda^L(z)\lambda^L(w) &\sim -(z-w)^{-1}\,, \\ \beta_i(z)\gamma_j(w) &\sim -2\delta_{ij}(z-w)^{-1}\,, & \sigma_i(z)\psi_j^L(w) &\sim -2\delta_{ij}(z-w)^{-1}\,. \end{aligned} \tag{4.17}$$

In order for the screening currents to have conformal weight one, we need that

$$Q = \frac{2}{\alpha_+}\,. \tag{4.18}$$

Furthermore, the screening charges indeed preserve the complete $N = 1$ structure, because we have the OPE

$$e^{-2\phi/\alpha_+}\left(\beta_i - \frac{2}{\alpha_+}\lambda^L\sigma_i\right)(z)\;\; G(w) \sim \frac{-e^{-2\phi/\alpha_+}\sigma_i(w)}{(z-w)^2} + \text{regular} \tag{4.19}$$

5. Spacetime supersymmetry

We next discuss spacetime supersymmetry of the $N = 1$ supersymmetric AdS sigma models. We take the compactification with $AdS_{2d_1+1} \times S^{2d_2+1}$. What we mean by S^{2d_2+1} is explained in section 2.2. The notation that we will use is as follows: variables and fields that describe AdS_{2d_1+1} will be denoted by the same symbols as in the previous section, the variables and fields from the second factor S^{2d_2+1} will be denoted by the same symbols as well, but with a tilde on them. The fields and quantities with a tilde have not yet been analytically continued. In particular, to get the right value for the central charge $c = 15$, we need $d_1 + d_2 + 1 = 5$ and $\alpha_+ = \pm i\tilde{\alpha}_+$. The zero modes of the fermions in the RR sector form some kind of Clifford algebra. We have for instance

$$\left\{(\sigma_i)_0, (\psi_j^L)_0\right\} = -2\delta_{ij}\,, \qquad \left\{\lambda_0^L, \lambda_0^L\right\} = -1\,. \tag{5.1}$$

Thus the zero modes can be represented via gamma matrices. We denote by Γ_ψ the gamma matrix representing the zero mode of ψ. Now the RR vacua of the fermions are created by spin fields V_α, where α runs from 1 to $2^{d_1+d_2+1}$. We also have the OPE

$$\psi(z)V_\alpha(w) = \frac{(\Gamma_\psi)_\alpha^\beta V_\beta(w)}{(z-w)^{1/2}} + \cdots \tag{5.2}$$

An explicit representation for the gamma matrices can be given using matrices like

$$\Gamma \sim \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_i \otimes 1 \otimes \cdots \otimes 1\,, \qquad i = 1, 2\,. \tag{5.3}$$

As in the usual NSR string, we will attempt to construct the spacetime supersymmetry generators using RR ground states of conformal weight 5/8 and the bosonized superreparametrization ghosts. What makes life complicated is that the most general vertex operator that creates a RR ground state is not some linear combination of the V_α. This is because we have the fields γ_i, whose zero modes are well-defined. These zero modes can be multiplied arbitrarily with the RR vacua and the state will still be a RR vacuum. Thus the most general vertex operator that creates an RR ground state is

$$V \equiv f^\alpha(\gamma_i, \tilde{\gamma}_i) V_\alpha \,. \tag{5.4}$$

We now want to examine which vertex operators survive in the perturbed $N=1$ theory. For this we need to check that they commute with the screening charges, and that they have correct OPE's with the supersymmetry generators.

We start by looking at the OPE with the screening currents. The $e^{-2\phi/\alpha_+}$ factor doesn't do anything and can be ignored. What remains is

$$\left(\beta_i - \frac{2}{\alpha_+}\lambda^L \sigma_i\right)(z)\ V(w) \sim \frac{-2\frac{\partial}{\partial\gamma_i} f^\alpha(\gamma_i,\tilde{\gamma}_i) V_\alpha + \frac{2}{\alpha_+} f^\alpha(\gamma_i,\tilde{\gamma}_i)(\Gamma_{\lambda^L}\Gamma_{\sigma_i})_\alpha^\beta V_\beta}{(z-w)} + \cdots \tag{5.5}$$

Thus we find a set of first order differential equations for the functions f^α. Probably, these are equivalent to the Killing spinor equations in $AdS_{2d_1+1} \times S^{2d_2+1}$ with NS two-form and dilaton turned on. To examine the solutions, we differentiate once more and we find

$$\begin{aligned}
\frac{\partial^2 f_\alpha}{\partial\gamma_i\partial\gamma_j} &= \frac{1}{\alpha_+^2} f_\beta (\Gamma_{\lambda^L}\Gamma_{\sigma_j}\Gamma_{\lambda^L}\Gamma_{\sigma_i})_\alpha^\beta = \frac{1}{2\alpha_+^2} f_\beta (\Gamma_{\sigma_j}\Gamma_{\sigma_i})_\alpha^\beta \\
\frac{\partial^2 f_\alpha}{\partial\gamma_i\partial\tilde{\gamma}_j} &= \frac{1}{\alpha_+\tilde{\alpha}_+} f_\beta (\Gamma_{\tilde{\lambda}^L}\Gamma_{\tilde{\sigma}_j}\Gamma_{\lambda^L}\Gamma_{\sigma_i})_\alpha^\beta \\
\frac{\partial^2 f_\alpha}{\partial\tilde{\gamma}_i\partial\tilde{\gamma}_j} &= \frac{1}{\tilde{\alpha}_+^2} f_\beta (\Gamma_{\tilde{\lambda}^L}\Gamma_{\tilde{\sigma}_j}\Gamma_{\tilde{\lambda}^L}\Gamma_{\tilde{\sigma}_i})_\alpha^\beta = \frac{1}{2\tilde{\alpha}_+^2} f_\beta (\Gamma_{\tilde{\sigma}_j}\Gamma_{\tilde{\sigma}_i})_\alpha^\beta \,.
\end{aligned}$$

Because the different Γ's anticommute, this leads to the consistency conditions

$$f_\beta(\Gamma_{\sigma_j}\Gamma_{\sigma_i})_\alpha^\beta = f_\beta(\Gamma_{\tilde{\sigma}_j}\Gamma_{\tilde{\sigma}_i})_\alpha^\beta = 0 \,. \tag{5.6}$$

The number of solutions to these equations is counted as follows: f_β can either be in the kernel of all Γ_{σ_i} at the same time, or it can be in the kernel of all Γ_{σ_i} except one. using the explicit representation of the Γ matrices above, one sees that there are no other possibilities, and that each of these possibilities reduces the numbers of supersymmetries by 2^{-d_1}. There are d_1+1 possibilities though. In a similar way the tilded sector can be analyzed. The result is that the number of RR ground states that commute with all screening charges is equal to

$$2^{-d_1}(d_1+1)2^{-d_2}(d_2+1)2^{d_1+d_2+1} = 2(d_1+1)(d_2+1) \,. \tag{5.7}$$

For AdS_3 we can make a change of variables

$$\hat{\sigma}_1 = \sigma_1, \quad \hat{\lambda}_L = \lambda_L - \frac{1}{\alpha_+}\gamma_1\sigma_1, \quad \hat{\psi}_1^L = \psi_1^L + \frac{2}{\alpha_+}\gamma_1\lambda^L - \frac{1}{\alpha_+^2}\gamma_1^2\sigma_1, \quad \hat{\beta}_1 = \beta_1 - \frac{2}{\alpha_+}\lambda^L\sigma_1 \,. \tag{5.8}$$

This change of variables leaves the OPE's of the fermions invariant, and it also leaves the OPE's between β and the fermions invariant (i.e. they commute). The screening current is now just $e^{-2\phi/\alpha_+}\hat{\beta}$, and clearly leaves all RR ground states made from hatted fermions invariant. This is in agreement with the result above where we found 8 invariant states for $AdS_3 \times AdS_3$.

It remains to examine whether the vertex operators that create the RR ground states have the correct OPEs with the $N = 1$ supersymmetry generator. To get no pole in the OPE of G with V of order 3/2, we need

$$\frac{\partial f^\beta}{\partial \gamma_i}(\Gamma_{\psi_i^L})^\alpha_\beta + \frac{Q}{2} f^\beta (\Gamma_{\lambda^L})^\alpha_\beta + \frac{\partial f^\beta}{\partial \tilde{\gamma}_i}(\Gamma_{\tilde{\psi}_i^L})^\alpha_\beta + \frac{\tilde{Q}}{2} f^\beta (\Gamma_{\tilde{\lambda}^L})^\alpha_\beta = 0 \tag{5.9}$$

Using the previous results, we can rewrite this as

$$\frac{1}{\alpha_+}\left(f^\beta(\Gamma_{\lambda^L}\Gamma_{\sigma_i}\Gamma_{\psi_i^L})^\alpha_\beta + f^\beta(\Gamma_{\lambda^L})^\alpha_\beta\right) = -\frac{1}{\tilde{\alpha_+}}\left(f^\beta(\Gamma_{\tilde{\lambda}^L}\Gamma_{\tilde{\sigma}_i}\Gamma_{\tilde{\psi}_i^L})^\alpha_\beta + f^\beta(\Gamma_{\tilde{\lambda}^L})^\alpha_\beta\right). \tag{5.10}$$

We already know that all Γ_{σ_i} except at most one yield zero when acting on f^β, and similarly for $\Gamma_{\tilde{\sigma}_i}$. Take an f^β for which all $\Gamma's$ except maybe Γ_{σ_1} and $\Gamma_{\tilde{\sigma}_1}$ have zero eigenvalue. We can then rewrite (5.10) as

$$f^\beta\left(\Gamma_\lambda[\Gamma_{\sigma_1}, \Gamma_{\psi_1^L}]\right)^\alpha_\beta = -\frac{\alpha_+}{\tilde{\alpha}_+} f^\beta\left(\Gamma_{\tilde{\lambda}}[\Gamma_{\tilde{\sigma}_1}, \Gamma_{\tilde{\psi}_1^L}]\right)^\alpha_\beta. \tag{5.11}$$

After some further manipulations we get

$$f^\beta\left([\Gamma_\lambda, \Gamma_{\tilde{\lambda}}][\Gamma_{\sigma_1}, \Gamma_{\psi_1^L}][\Gamma_{\tilde{\sigma}_1}, \Gamma_{\tilde{\psi}_1^L}]\right)^\alpha_\beta = 8\frac{\tilde{\alpha}_+}{\alpha_+} f^\alpha \,. \tag{5.12}$$

The operator $[\Gamma_\lambda, \Gamma_{\tilde{\lambda}}]$ has eigenvalues $\pm 2i$, whereas $[\Gamma_{\sigma_1}, \Gamma_{\psi_1^L}]$ and $[\Gamma_{\tilde{\sigma}_1}, \Gamma_{\tilde{\psi}_1^L}]$ have eigenvalue ± 2. Thus we see that the eigenvalues are correlated in some way, depending on whether $\tilde{\alpha}_+ = +i\alpha_+$ or $-i\alpha_+$. (which was needed in order that we get the central charge $c = 3(d_1+d_2+1)$). This correlation, which also applies to the the cases when other Γ_{σ_i} annihilate f^β, removes another half of the supersymmetries, leaving us with $(d_1 + 1)(d_2 + 1)$ spacetime supersymmetries. (Of course, in case $d_1 + d_2 < 4$ we still need to add an internal sector to the theory which may increase the number of supersymmetries. This only happens for $d_1 + d_2 = 2$ however, in which case we can get an extra factor of two.) Finally, the total number of space-time supersymmetries after combining left- and right movers is twice the result above, i.e. $2(d_1+1)(d_2+1)$.

The correlation of the signs found above can be naturally embedded in some GSO projection; there is a subtlety related to the fact that we have γ-dependent coefficients. However, differentiating with respect to some γ acts as two gamma matrices and will not disturb the sign correlation found above.

Let us illustrate this for $AdS_5 \times S^5$. This is a theory with ten bosons and fermions and central charge $c = 15$, and can therefore be used as an NSR string background. Denote the RR ground states as usual with five $\pm$ signs. The first two refer to the eigenvalues of $[\Gamma_{\sigma_i}, \Gamma_{\psi_i^L}]$, the third one to the eigenvalue of $[\Gamma_\lambda, \Gamma_{\tilde{\lambda}}]$ and the last two to the eigenvalues of $[\Gamma_{\tilde{\sigma}_i}, \Gamma_{\tilde{\psi}_i^L}]$. Then the nine ground states correspond to those combinations $(\pm,\pm,\pm,\pm,\pm)$ for which the first two signs are not both minus, the last two signs are not both minus, and for which the product of all signs is $+1$. The real RR ground states are, however, still linear combinations of such states with γ-dependent coefficients.

The $AdS_5 \times S^5$ theory has a rather rich structure. For example, the following five currents are conserved (i.e. commute with all the screening charges)

$$\begin{aligned}
J_1 &= i\left(\lambda - \frac{1}{\alpha_+}\sigma_r\gamma_r\right)\left(\tilde{\lambda} - \frac{1}{\tilde{\alpha}_+}\tilde{\sigma}_r\tilde{\gamma}_r\right) \\
J_2 &= i\left(\partial\phi - \frac{1}{\alpha_+}\beta_r\gamma_r + \frac{1}{\alpha_+}\sigma_r\psi_r^L\right) \\
J_3 &= i\left(\partial\tilde{\phi} - \frac{1}{\tilde{\alpha}_+}\tilde{\beta}_r\tilde{\gamma}_r + \frac{1}{\tilde{\alpha}_+}\tilde{\sigma}_r\tilde{\psi}_r^L\right) \\
J_4 &= \frac{1}{2}\left(\sigma_1\psi_1^L - \sigma_2\psi_2^L - \frac{2}{\alpha_+}\lambda\sigma_1\gamma_1 + \frac{2}{\alpha_+}\lambda\sigma_2\gamma_2 - \frac{2}{\alpha_+^2}\sigma_1\sigma_2\gamma_1\gamma_2\right) \\
J_5 &= \frac{1}{2}\left(\tilde{\sigma}_1\tilde{\psi}_1^L - \tilde{\sigma}_2\tilde{\psi}_2^L - \frac{2}{\tilde{\alpha}_+}\tilde{\lambda}\tilde{\sigma}_1\tilde{\gamma}_1 + \frac{2}{\tilde{\alpha}_+}\tilde{\lambda}\tilde{\sigma}_2\tilde{\gamma}_2 - \frac{2}{\tilde{\alpha}_+^2}\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\gamma}_1\tilde{\gamma}_2\right).
\end{aligned} \tag{5.13}$$

In J_2 and J_3 we recognize a supersymmetric extension of the current J_0 discussed in section 2.3. Out of these five currents one can construct the following linear combination

$$J_{\mathrm{U}(1)} = J_1 + \frac{2}{\tilde{\alpha}_+}J_2 - \frac{2}{\alpha_+}J_3 + J_4 + J_5\,. \tag{5.14}$$

If we decompose the $N = 1$ generator G in terms of its $+1$ and -1 eigenvalue with respect to $J_{\mathrm{U}(1)}$, we find that $J_{\mathrm{U}(1)}, G^+, G^-, T$ generate the $c = 15$ $N = 2$ algebra. This $N = 2$ algebra will probably play a crucial role in the $AdS_5 \times S^5$ theory, in particular in finding the precise GSO projection that will give rise to a modular invariant string theory.

6. WZW cosets for supergroups

It is very interesting to repeat the construction of section 3 for supergroups, especially in view of the results of [10] and [14]. Because supergroups contain anticommmuting world-sheet scalars, at first sight one expects to obtain sigma models in the GS formalism. However, we were unable to find κ symmetry in the examples below, and furthermore the results of [10] indicate that this is probably not the correct interpretation.

Below we will discuss two examples of super"cosets". Although the precise embedding of these theories in a critical string theory remains elusive, both theories have several interesting features. In particular, both have as target space $AdS_5 \times S^5$ (after analytic continuation). The first example is the coset SL(3|3)/SL(2)^2, the second one the coset SL(4|4)/SP(2)^2. The first one has 18 anticommuting scalars, suggesting a close relation with the NSR model of section 4, the second one has 32 anticommuting scalars, suggesting a close relation with the usual GS string. The treatment of SL(4|4)/SP(2)^2 will not be quite as satisfactory as that of SL(3|3)/SL(2)^2, because we have to adjust the levels of the two SL(4)$'s$ in SL(4|4) at an intermediate stage, but the final result will still be an exact CFT.

6.1 SL(3|3)/SL(2)^2

We basically want to follow the same procedure as we did for SL(3)/SL(2) in section 3. We again write $g = g_u g_{\mathrm{SL}(2)^2} g_D g_l$. To define the various group elements appearing in this decomposition we introduce the 3×3 matrices

$$A_1 = \begin{pmatrix} 1 & \bar{\gamma}_1 & \bar{\gamma}_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

$$A_2 = \begin{pmatrix} 1 & 0 & \bar{\gamma}_4 \\ 0 & 1 & \bar{\gamma}_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.2}$$

$$D_1 = \begin{pmatrix} e^{2\phi_1} & 0 & 0 \\ 0 & e^{-\phi_1} & 0 \\ 0 & 0 & e^{-\phi_1} \end{pmatrix} \tag{6.3}$$

$$D_2 = \begin{pmatrix} e^{\phi_2} & 0 & 0 \\ 0 & e^{\phi_2} & 0 \\ 0 & 0 & e^{-2\phi_2} \end{pmatrix} \tag{6.4}$$

$$\Psi = \begin{pmatrix} \bar{\psi}_{11} & \bar{\psi}_{12} & \bar{\psi}_{13} \\ \bar{\psi}_{21} & \bar{\psi}_{22} & \bar{\psi}_{23} \\ \bar{\psi}_{31} & \bar{\psi}_{32} & \bar{\psi}_{33} \end{pmatrix} \tag{6.5}$$

$$g_u = \begin{pmatrix} A_1 & \Psi \\ 0 & A_2 \end{pmatrix} \tag{6.6}$$

$$g_d = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \tag{6.7}$$

$$g_l = \begin{pmatrix} A_1^\dagger & 0 \\ \Psi^\dagger & A_2^\dagger \end{pmatrix} . \tag{6.8}$$

The two SL(2)'s live in the bottom right 2×2 block of D_1, and the top left 2×2 block of D_2. After removing the $\mathrm{SL}(2)^2$ degrees of freedom, the remaining action is given by the WZW action evaluated on $g = g_u g_D g_l$. In addition, we need a linear dilaton background in order to make the theory exactly conformal. We will determine this dilaton background later. The WZW action evaluated for $g = g_u g_D g_l$ reads

$$\begin{aligned} S = & \frac{3k}{2\pi}\int \left(\partial\phi_1\bar\partial\phi_1 - \partial\phi_2\bar\partial\phi_2\right) + \\ & + \frac{k}{2\pi}\int \mathrm{tr}\left(\left(\partial A_1 D_1 \bar\partial A_1^\dagger D_1^{-1}\right) - \left(\partial A_2 D_2 \bar\partial A_2^\dagger D_2^{-1}\right)\right) + \\ & + \frac{k}{2\pi}\int \mathrm{tr}\left(\left(A_1 D_1 A_1^\dagger\right)^{-1} \partial(\Psi A_2^{-1})\left(A_2 D_2 A_2^\dagger\right)\bar\partial\left(\left(\Psi A_2^{-1}\right)^\dagger\right)\right) . \end{aligned} \tag{6.9}$$

Upon expanding the first two lines in $\phi_i, \chi_i, \bar\chi_i$ we see that they describe two copies of the AdS_5 sigma model discussed in section 2. The last line in (6.9) is a term bilinear in the anticommuting scalars $\Psi, \Psi^\dagger$. The signs arise because the action for SL(3|3) contains a supertrace rather than an ordinary trace. Furthermore we notice that the first two lines of (6.9) can be combined to form $kW(A_1D_1A_1^\dagger) - kW(A_2D_2A_2^\dagger)$, with W the WZW action for SL(4).

The next step in the construction is to bosonize (6.9). We will do this in two steps. The first step involves the anticommuting world-sheet scalars $\Psi, \Psi^\dagger$, the second step involves only the bosons of the theory.

To perform the first step we start by redefining the field Ψ as $\Psi A_2^{-1} = \tilde\Theta$ in order to simplify the last term in (6.9). Next we redefine $\tilde\Theta = M_1\Theta M_2^{-1}$ with $M_1 = A_1D_1A_1^+, M_2 = A_2D_2A_2^+$. This changes the last term i (6.9) into:

$$\left(\partial\Theta + M_1^{-1}\partial M_1\Theta - \Theta M_2^{-1}\partial M_2\right)\bar\partial\Theta^+ . \tag{6.10}$$

This can be transformed in the action for a fermionic β,γ-system by introducing the new field

$$\beta_\Theta = \partial\Theta + M_1^{-1}\partial M_1\Theta - \Theta M_2^{-1}\partial M_2 . \tag{6.11}$$

From this expression we see that Θ transforms under the action of $\mathrm{SL}(3)\times\mathrm{SL}(3)$ into $M_1^{-1}\Theta M_2$ and therefore the jacobian for the change of variables (6.11) is given by the WZW actions for M_1 and M_2, computed in the bi-fundamental representation. Since the bi-fundamental representation of $\mathrm{SL}(3)\times\mathrm{SL}(3)$ contains 3 fundamental representations of each SL(3), we obtain the following correction to the action

$$+3W((A_1D_1A_1^\dagger)) + 3W(A_2D_2A_2^\dagger) . \tag{6.12}$$

This, combined with the first two terms in our action, leads to a renormalisation of the coupling in front of these terms. The full action at this stage reads

$$(k+3)W((A_1D_1A_1^\dagger)) + (-k+3)W(A_2D_2A_2^\dagger) + \frac{k}{2\pi}\int \beta_\Theta\bar\partial\Theta^\dagger . \tag{6.13}$$

Having bosonized the anti-commuting degrees of freedom, it remains to bosonize the bosonic degrees of freedom. This can be done exactly as explained in section 3. The first two terms in (6.13) correspond to two SL(3)/SL(2) versions of our "coset" construction, one with level $k+3$ and another with level $-k+3$. Thus we can use the results in section 3.2 and conclude that the shift by 3 gets canceled. We also obtain two bosonic screening charges for each SL(3) of the form given in (3.16). After a rescaling of the fields the system is described by the action

$$S = \frac{1}{4\pi}\int \Big(\big(\partial\phi_1\bar{\partial}\phi_1 - \partial\phi_2\bar{\partial}\phi_2\big) + Q_1 R\phi_1 + Q_2 R\phi_2 + \beta_i^1\bar{\partial}\gamma_i^1 + \beta_1^2\bar{\partial}\gamma_i^2 + \beta_\Theta\bar{\partial}\Theta^+ \Big)\,, \tag{6.14}$$

with

$$Q_1 = \sqrt{\frac{3}{8k}}\,, \qquad Q_2 = \sqrt{\frac{3}{8k}}\,. \tag{6.15}$$

The change of variables for both the anti-commuting and the commuting degrees of freedom generate screening charges. The bosonic ones are obtained from (3.16), and in the normalization (6.14) they are given by

$$\begin{aligned} S_i &= \int \beta_i e^{\sqrt{3/2k}\phi_1}\,, \qquad i = 1, 2\,, \\ S_i &= \int \beta_i e^{\sqrt{3/2k}\phi_2}\,, \qquad i = 3, 4\,. \end{aligned} \tag{6.16}$$

These screening charges do not have conformal weight equal to zero, because the charges Q_i are equal to twice the momenta in (6.16). Therefore this theory is not conformal. This should not come as a surprise, because we already knew from the SL(3)/SL(2) example in section 2 that the only way to make sense out of the theory is add a suitable linear dilaton background from the start. Here we see that we should have added the linear dilaton background

$$\Phi = -\frac{Q_1}{2}\phi_1 - \frac{Q_2}{2}\phi_2 \tag{6.17}$$

at the beginning to make the theory conformal. The final exact CFT is therefore given by (6.14) with Q_i replaced by $Q_i/2$, it has central charge

$$c = 5 + 12\frac{3}{32k} + 5 - 12\frac{3}{32k} - 18 = -8 \tag{6.18}$$

and it has the screening charges (6.16), plus certain fermionic screening charges. The latter originate from the change of variables (6.11):

$$M_1\beta_\Theta M_2^{-1} = \partial(M_1\Theta M_2^{-1})\,. \tag{6.19}$$

Some of the fermionic screening charges are simple, and some of them are iterated. Following the discussion in section 3.2, we will only present the simple fermionic screening charges and verify whether they have conformal weight one. This is a non-trivial statement, as we already used all available freedom to change the background charges of the scalars in order to give the bosonic screening charges the right conformal weight.

The fermionic screening charges, as is clear from (6.19), are the contour integrals of $M_1\beta_\Theta M_2^{-1}$. Let us denote the components of β_Θ by β_{ij}, and the i,j component of $M_1\beta_\Theta M_2^{-1}$ by S_{ij}. Then out of a total of nine fermionic screening currents we find the following four simple ones charges

$$\begin{aligned} S_{21} &= e^{(-\phi_1-\phi_2)/\sqrt{6k}}(\beta_{21}+\beta_{11}\gamma_1-\beta_{23}\gamma_4-\beta_{13}\gamma_1\gamma_4) \\ S_{22} &= e^{(-\phi_1-\phi_2)/\sqrt{6k}}(\beta_{22}+\beta_{12}\gamma_1-\beta_{23}\gamma_3-\beta_{13}\gamma_1\gamma_3) \\ S_{31} &= e^{(-\phi_1-\phi_2)/\sqrt{6k}}(\beta_{31}+\beta_{11}\gamma_2-\beta_{33}\gamma_4-\beta_{13}\gamma_2\gamma_4) \\ S_{32} &= e^{(-\phi_1-\phi_2)/\sqrt{6k}}(\beta_{32}+\beta_{12}\gamma_2-\beta_{33}\gamma_3-\beta_{13}\gamma_2\gamma_3) \end{aligned} \tag{6.20}$$

The conformal weight of $\exp(p_1\phi_1+p_2\phi_2)$ equals $-p_1(p_1+Q_1)/2+p_2(p_2+Q_2)/2$, and since $Q_1=Q_2$ this vanishes when $p_1=p_2$. Therefore all four fermionic screening charges have the right conformal weight.

This completes the construction of the CFT announced in the introduction for the case of $\mathrm{SL}(3|3)/\mathrm{SL}(2)^2$. It has $AdS_5\times S^5$ target space, 18 anticommuting world-sheet scalars and bosonic and fermionic B-fields turned on. We have shown that theory is exactly conformal and can be bosonised with the help of the free fields and screening operators listed above.

6.2 A Comment on the Bosonization of $\mathrm{SL}(N|N)$

The change of variables (6.11) can also be applied to the $\mathrm{SL}(N|N)$ WZW theory. The result is an action of the form

$$S=(k+N)W(g_1)+(-k+N)W(g_2)+\frac{k}{2\pi}\int d^2z\beta_\Theta\bar\partial\Theta^\dagger \tag{6.21}$$

where g_1 and g_2 are arbitrary sl_3 group elements, and β_Θ,Θ are anticommuting $N\times N$ matrices. Explicit expressions for the $\mathrm{SL}(N|N)$ currents in terms of the variables in (6.21) are given in [17]. The central charge of (6.21) is

$$c=\frac{(k+N)(N^2-1)}{k+N-N}+\frac{(-k+N)(N^2-1)}{-k+N-N}-2N^2=-2\,, \tag{6.22}$$

which is indeed the correct result for $\mathrm{SL}(N|N)$. By adding two more free scalar fields we find a theory that has $c=0$. this is the $\mathrm{GL}(N|N)$ theory that was studied as a topological field theory in [28]. The change of variables above maps it to the topological $\mathrm{GL}(N)/\mathrm{GL}(N)$ theory, although this is not an exact equivalence, because the change of variables introduces certain fermionic screening charges in the theory. Along the same lines, the $\mathrm{SL}(N|N)$ WZW theory is can be mapped to the topological $\mathrm{SL}(N)/\mathrm{SL}(N)$ theory together with one additional free anticommuting β,Θ system.

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In the case of SL(2|2), the map above is essentially the one discussed in section 10 of [10], namely it maps the SL(2|2) WZW model to two copies of the SL(2) WZW model plus four anticommuting $\beta_\Theta, \Theta^\dagger$ systems. A further change of variables then maps it to the standard NSR description of strings on $AdS_3 \times S^3$ with only NS background fields turned on.

6.3 $\mathrm{SL}(4,4)/\mathrm{SP}(2)^2$

Because the GS string on $AdS_5 \times S^5$ is described by a sigma model whose target space is (a suitable real form of) the coset superspace $\mathrm{SL}(4,4)/\mathrm{SP}(2)^2$, it is very interesting to examine what happens when we examine our "coset" for $\mathrm{SL}(4,4)/\mathrm{SP}(2)^2$. One novelty here is that we don't know whether it is possible to express SL(4) currents in terms of SP(2) currents and additional free scalars and β, γ systems. This is because SP(2) is not generated by a subset of the positive and negative simple roots, and therefore the results in [18] cannot be used. Still, we will try to construct an exact CFT by repeating the procedure we have been following so far. We write $g = g_u g_{\mathrm{SP}(2)^2} g_d g_l$, and then remove the $\mathrm{SP}(2)^2$ degrees of freedom. It will turn out that in order to obtain an exact CFT, we not only need to adjust the background charges, but we also need to adjust the levels of some of the WZW actions that appear in the discussion. This is probably related to the absence of a realization of SL(4) currents in terms of SP(2), as mentioned above.

Except for this, the construction of the exact CFT follows closely the construction for $\mathrm{SL}(3|3)/\mathrm{SL}(2)^2$. We begin by defining g_u, g_d, g_l in terms of the auxiliary 4×4 matrices

$$A_1 = \begin{pmatrix} 1 & \bar{\gamma}_1 & \bar{\gamma}_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.23}$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \bar{\gamma}_4 \\ 0 & 0 & 1 & \bar{\gamma}_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.24}$$

$$D_1 = \begin{pmatrix} e^{3\phi_1} & 0 & 0 & 0 \\ 0 & e^{-\phi_1} & 0 & 0 \\ 0 & 0 & e^{-\phi_1} & 0 \\ 0 & 0 & 0 & e^{-\phi_1} \end{pmatrix} \tag{6.25}$$

$$D_2 = \begin{pmatrix} e^{\phi_2} & 0 & 0 & 0 \\ 0 & e^{\phi_2} & 0 & 0 \\ 0 & 0 & e^{\phi_2} & 0 \\ 0 & 0 & 0 & e^{-3\phi_2} \end{pmatrix} \tag{6.26}$$

$$\Psi = \begin{pmatrix} \bar{\psi}_{11} & \bar{\psi}_{12} & \bar{\psi}_{13} & \bar{\psi}_{14} \\ \bar{\psi}_{21} & \bar{\psi}_{22} & \bar{\psi}_{23} & \bar{\psi}_{24} \\ \bar{\psi}_{31} & \bar{\psi}_{32} & \bar{\psi}_{13} & \bar{\psi}_{34} \\ \bar{\psi}_{41} & \bar{\psi}_{42} & \bar{\psi}_{43} & \bar{\psi}_{44} \end{pmatrix}. \tag{6.27}$$

The WZW action $kW(g_u g_d g_l)$ is evaluated by taking

$$g_u = \begin{pmatrix} A_1 & \Psi \\ 0 & A_2 \end{pmatrix} \tag{6.28}$$

$$g_d = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \tag{6.29}$$

$$g_l = \begin{pmatrix} A_1^\dagger & 0 \\ \Psi^\dagger & A_2^\dagger \end{pmatrix} \tag{6.30}$$

and this leads to an action which is very similar to (6.9):

$$\begin{aligned} S = & \frac{3k}{\pi} \int \left(\partial\phi_1 \bar{\partial}\phi_1 - \partial\phi_2 \bar{\partial}\phi_2 \right) + \\ & + \frac{k}{2\pi} \int \mathrm{tr} \left(\left(\partial A_1 D_1 \bar{\partial} A_1^\dagger D_1^{-1} \right) - \left(\partial A_2 D_2 \bar{\partial} A_2^\dagger D_2^{-1} \right) \right) + \\ & + \frac{k}{2\pi} \int \mathrm{tr} \left(\left(A_1 D_1 A_1^\dagger \right)^{-1} \partial(\Psi A_2^{-1}) \left(A_2 D_2 A_2^\dagger \right) \bar{\partial}((\Psi A_2^{-1})^\dagger) \right). \end{aligned} \tag{6.31}$$

Again, the first two lines are just $kW(A_1 D_1 A_1^\dagger) - kW(A_2 D_2 A_2^\dagger)$, with W the WZW action. However, in contrast to the case of $\mathrm{SL}(3|3)/\mathrm{SL}(2)^2$, it turns out that the classical action (6.9) does not quite correspond to an exact CFT. We have to slightly modify it by hand, namely in order to obtain an exact CFT we should adjust the levels in (6.31) to

$$\begin{aligned} S = & \left(k - \frac{4}{3} \right) W(A_1 D_1 A_1^\dagger) + \left(-k - \frac{4}{3} \right) W(A_2 D_2 A_2^\dagger) + \\ & + \frac{k}{2\pi} \int \mathrm{tr} \left((A_1 D_1 A_1^\dagger)^{-1} \partial(\Psi A_2^{-1}) (A_2 D_2 A_2^\dagger) \bar{\partial}((\Psi A_2^{-1})^\dagger) \right). \end{aligned} \tag{6.32}$$

These shifts of k look rather awkward. Perhaps there is a better way to obtain the same final answer without making such shifts in the levels. At this point we lack a better geometrical description of this "coset". We will comment on this later.

To continue the bosonization procedure, we redefine the field Ψ exactly as in (6.10) and (6.11). In this case, Θ transforms in the bifundamental representation of $\mathrm{SL}(4) \times \mathrm{SL}(4)$ which contains four fundamentals for each $\mathrm{SL}(4)$. This leads to a correction to the action of the form

$$+4W(A_1 D_1 A_1^\dagger) + 4W(A_2 D_2 A_2^\dagger) \tag{6.33}$$

and the full action at this stage reads

$$\left(k + \frac{8}{3} \right) W(A_1 D_1 A_1^\dagger) + \left(-k + \frac{8}{3} \right) W(A_2 D_2 A_2^\dagger) + \frac{k}{2\pi} \int \beta_\Theta \bar{\partial} \Theta^\dagger . \tag{6.34}$$

Next, we bosonize the bosonic part of the action. This is given by the SL(4)/SP(2) version of our general construction. This is similar to SL(4)/SL(3), except that there are only two rather than three $\gamma, \bar{\gamma}$ pairs. Correspondingly, the shift of k will not be by -4 but only by $-8/3$.[3] As a result, the various 8/3's cancel, and after a suitable rescaling of the fields we obtain the free field action

$$S = \frac{1}{4\pi} \int \Big(\big(\partial\phi_1 \bar{\partial}\phi_1 - \partial\phi_2 \bar{\partial}\phi_2 \big) + Q_1 R\phi_1 + Q_2 R\phi_2 + \\ + \beta_i^1 \bar{\partial}\gamma_i^1 + \beta_1^2 \bar{\partial}\gamma_i^2 + \beta_\Theta \bar{\partial}\Theta^+ \Big) , \tag{6.35}$$

with background charges

$$Q_1 = \frac{4}{\sqrt{3k}}, \qquad Q_2 = \frac{4}{\sqrt{3k}} . \tag{6.36}$$

Now everything is practically the same as before. The various changes of variables give rise to bosonic and fermionic screening charges. The explicit expressions for the simple screening charges will be given in a moment. In order that the screening charges have conformal weight zero, we need to replace Q_i by $Q_i/2$ in (6.35). Thus the final exact CFT is given by (6.35), with $Q_i = 2/\sqrt{3k}$. It has central charge -22, four bosonic screening charges and sixteen fermionic screening charges, of which nine are simple and seven are iterated. The bosonic screening charges are

$$\begin{aligned} S_i &= \int \beta_i e^{2\phi_1/\sqrt{3k}} , \qquad & i = 1, 2 , \\ S_i &= \int \beta_i e^{2\phi_2/\sqrt{3k}} , \qquad & i = 3, 4 . \end{aligned} \tag{6.37}$$

To write down the simple fermionic screening charges, we denote the components of β_Θ by β_{ij}, and the i, j component of $M_1 \beta_\Theta M_2^{-1}$ by S_{ij}. The nine simple fermionic screening currents are given by

$$\begin{aligned} S_{21} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{21} + \beta_{11}\gamma_1) \\ S_{22} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{22} + \beta_{12}\gamma_1 - \beta_{24}\gamma_4 - \beta_{14}\gamma_1\gamma_4) \\ S_{23} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{23} + \beta_{13}\gamma_1 - \beta_{24}\gamma_3 - \beta_{14}\gamma_1\gamma_3) \\ S_{31} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{31} + \beta_{11}\gamma_2) \\ S_{32} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{32} + \beta_{12}\gamma_2 - \beta_{34}\gamma_4 - \beta_{14}\gamma_2\gamma_4) \\ S_{33} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{33} + \beta_{13}\gamma_2 - \beta_{34}\gamma_3 - \beta_{14}\gamma_2\gamma_3) \\ S_{41} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} \beta_{41} \\ S_{42} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{42} - \beta_{44}\gamma_4) \\ S_{43} &= e^{(-\phi_1-\phi_2)/\sqrt{12k}} (\beta_{43} - \beta_{44}\gamma_3) . \end{aligned} \tag{6.38}$$

This concludes the description of the exact CFT based on $SL(4|4)/SP(2)^2$.

[3]The fact that the shift here is $-8/3$ as opposed to -4 explains the adjustment we had to make in order to go from (6.31) to (6.32). If we would be able to express SL(4) currents in terms of SP(2) currents and free ϕ, β, γ fields, there probably would be no need to adjust levels by hand and the whole construction would appear more natural. However, at present we do not know either algebraically or in the Lagrangian approach whether this is possible.

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7. Conclusions

In this paper we considered various conformal field theories whose target space geometry is an anti-de Sitter space. We have not yet achieved a full understanding of these CFT's. One issue that requires a precise justification is that of simple versus iterated screening charges. We assumed by analogy with WZW theory (when the rank of the group is greater than one) that the conformal blocks of the NSR version of the AdS_{2d+1} sigma model can be computed using only simple screening charges. A deeper understanding of this assumption and the way the left and right movers should be combined is certainly desirable, in particular in order to construct a complete modular invariant theory with a proper GSO projection. In this context we would like to note the similarities with $c < 1$ theories (Minimal Models) coupled to $c > 1$ Liouville theory, where the non-compact part of our target space plays the role of the Liouville model and the compact part (i.e. the part which is analytically continued in ϕ) corresponds to the Minimal Model. However, the quantization of strings on AdS spaces will undoubtedly lead to the same problems that one encounters when one studies string theory on $\mathrm{SL}(2,\mathbb{R})$ [29].

Although the coset theories for supergroups discussed in section 6 look very suggestive, it is not yet clear how to use them to construct consistent string backgrounds. Based on the number of supersymmetries, the $\mathrm{SL}(3|3)/\mathrm{SL}(2)^2$ model appears related to the NSR string on $AdS_5 \times S^5$ of section 4, whereas the $\mathrm{SL}(4|4)/\mathrm{SP}(2)^2$ appears related to the "usual" string theory on $AdS_5 \times S^5$ with RR five-form flux. It would be very interesting to understand these relations in more detail. In particular, one could look for additional world-sheet symmetries, or for an analogue of κ-symmetry. Once the space-time interpretation of these theories is understood, one could analyze whether they do in fact describe theories with some RR background turned on (as we already mentioned, we suspect that RR backgrounds are related to fermionic screening charges) or whether one needs to perturb the theories away from the "WZW" point in order to turn on RR backgrounds.

The theories we have described in this paper do have holographic behavior, and this raises the question which boundary quantum field theories are associated to them. To answer this question, it would be helpful to know to which brane configurations they correspond. The $AdS_{2d_1+1} \times S^{2d_2+1}$ solutions appear to be related to peculiar delocalized superpositions of intersecting fundamental strings and NS fivebranes. The meaning of these configurations is, however, rather obscure.

To summarize, we have described various exact CFT's with AdS target spaces, some of which can be used to build critical string backgrounds. We believe that a further study of these theories will teach us more about exactly solvable CFT's and their space-time interpretation, as well as the world-sheet formulation of AdS/CFT duality.

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